MR6, 2015 All solutions in one file.

J355. Proposed by Anant Mudgal, India

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$4(a^2+b^2+c^2)-(a^3+b^3+c^3)\geq 9.$$

Solution by Arkady Alt , San Jose ,California, USA.

Since $a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=9-2(ab+bc+ca)$
and $a^3+b^3+c^3=3abc+(a+b+c)^3-3(a+b+c)(ab+bc+ca)=$
$3abc+27-9(ab+bc+ca)$ then $4(a^2+b^2+c^2)-(a^3+b^3+c^3)\geq 9 \iff$
$4(9-2(ab+bc+ca))-(3abc+27-9(ab+bc+ca))\geq 9 \iff$
$36-8(ab+bc+ca)-3abc-27+9(ab+bc+ca)\geq 9 \iff$
$ab+bc+ca\geq 3abc \iff (ab+bc+ca)(a+b+c)\geq 9abc,$ where
latter inequality is right because by AM-GM inequality
$ab+bc+ca\geq 3\sqrt[3]{a^2b^2c^2}$ and $a+b+c\geq 3\sqrt[3]{abc}.$

◊

J356. Proposed by Titu Andreescu, University of Texas at Dallas, USA

Find all positive integers n such that

$$2(6+9i)^n-3(1+8i)^n=3(7+4i)^n.$$

Solution by Arkady Alt , San Jose ,California, USA.

Note that $2(6+9i)^n-3(1+8i)^n=3(7+4i)^n \iff$

$$2\cdot 3^{n-1}=\left(\frac{1+8i}{2+3i}\right)^n+\left(\frac{7+4i}{2+3i}\right)^n \iff (2+i)^n+(2-i)^n=2\cdot 3^{n-1}.$$

Also note that for any positive integers n such that
$(2+i)^n+(2-i)^n=2\cdot 3^{n-1}$ holds inequality
$2\cdot 3^{n-1}=|(2+i)^n+(2-i)^n|\leq|(2+i)^n|+|(2-i)^n|=|(2+i)|^n+|(2-i)|^n=2\left(\sqrt{5}\right)^n.$

But for any positive integers $n\geq 4$ holds inequality $3^{n-1}>\left(\sqrt{5}\right)^n.$
Indeed, $3^{4-1}>\left(\sqrt{5}\right)^4 \iff 27>25$ and for any $n\geq 4$ in supposition
$3^{n-1}>\left(\sqrt{5}\right)^n$ we obtain

$$3^n=3^{n-1}\cdot 3>3\left(\sqrt{5}\right)^n>\sqrt{5}\cdot\left(\sqrt{5}\right)^n=\left(\sqrt{5}\right)^{n+1}.$$

Thus, to find positive integers n that $(2+i)^n+(2-i)^n=2\cdot 3^{n-1}$
suffice to check $n\in\{1,2,3\}$.
For $n=1$ and $n=3$ we have, respectively, $(2+i)^1+(2-i)^1=4\neq 2\cdot 3^{1-1}$
and $(2+i)^3+(2-i)^3=4\neq 2\cdot 3^{3-1}.$
For $n=2$ we have $(2+i)^2+(2-i)^2=6=2\cdot 3^{2-1}.$
So, answer is $n=1$.

◊

J357. Proposed by Mihály Bencze, Braşov, România

Prove that for any $z\in\mathbb{C}$ such that $\left|z+\frac{1}{z}\right|=\sqrt{5},$

$$\left(\frac{\sqrt{5}-1}{2}\right)^2\leq|z|\leq\left(\frac{\sqrt{5}+1}{2}\right)^2.$$

Solution by Arkady Alt , San Jose ,California, USA.

Let $z = re^{i\varphi}$. Then $|z| = r$ and $\left|z + \dfrac{1}{z}\right|^2 = \left|re^{i\varphi} + \dfrac{1}{r}e^{-i\varphi}\right|^2 =$
$\dfrac{1}{r^2}\left|r^2 e^{2i\varphi} + 1\right|^2 = \dfrac{1}{r^2}\left|r^2\cos 2\varphi + 1 + ir^2 \sin 2\varphi\right|^2 =$
$\dfrac{\left(r^2\cos 2\varphi + 1\right)^2 + r^4\sin^2 2\varphi}{r^2} = \dfrac{r^4 + 2r^2\cos 2\varphi + 1}{r^2}$.

Hence, $\left|z + \dfrac{1}{z}\right| = \sqrt{5} \iff \left|z + \dfrac{1}{z}\right|^2 = 5 \iff r^4 + 2r^2\cos 2\varphi + 1 = 5r^2 \iff$
$2\cos 2\varphi = 5 - r^2 - \dfrac{1}{r^2}$ and since $|\cos 2\varphi| \le 1$ then $\left|5 - r^2 - \dfrac{1}{r^2}\right| \le 2 \iff$
$-2 \le 5 - r^2 - \dfrac{1}{r^2} \le 2 \iff \begin{cases} 0 \le r^4 - 3r^2 + 1 \\ r^4 - 7r^2 + 1 \le 0 \end{cases} \iff$

$$\begin{cases} r^2 \le \dfrac{3-\sqrt{5}}{2} \text{ or } \dfrac{3+\sqrt{5}}{2} \le r^2 \\ \dfrac{7-3\sqrt{5}}{2} \le r^2 \le \dfrac{7+3\sqrt{5}}{2} \end{cases} \iff \left[\begin{array}{l} \dfrac{7-3\sqrt{5}}{2} \le r^2 \le \dfrac{3-\sqrt{5}}{2} \\ \dfrac{3+\sqrt{5}}{2} \le r^2 \le \dfrac{7+3\sqrt{5}}{2} \end{array}\right. \iff$$

$$\left[\begin{array}{l} \left(\dfrac{3-\sqrt{5}}{2}\right)^2 \le r^2 \le \left(\dfrac{\sqrt{5}-1}{2}\right)^2 \\ \left(\dfrac{\sqrt{5}+1}{2}\right)^2 \le r^2 \le \left(\dfrac{3+\sqrt{5}}{2}\right)^2 \end{array}\right. \iff \left[\begin{array}{l} \dfrac{3-\sqrt{5}}{2} \le r \le \dfrac{\sqrt{5}-1}{2} \\ \dfrac{\sqrt{5}+1}{2} \le r \le \dfrac{3+\sqrt{5}}{2} \end{array}\right. \implies$$

$$\dfrac{3-\sqrt{5}}{2} \le r \le \dfrac{3+\sqrt{5}}{2} \iff \left(\dfrac{\sqrt{5}-1}{2}\right)^2 \le r \le \left(\dfrac{\sqrt{5}+1}{2}\right)^2.$$

Remark.

As we can see that although the numbers $\left(\dfrac{\sqrt{5}-1}{2}\right)^2$ and $\left(\dfrac{\sqrt{5}+1}{2}\right)^2$ are the minimum and maximum values of $|z|$ respectively, nonetheless the segment $\left[\left(\dfrac{\sqrt{5}-1}{2}\right)^2, \left(\dfrac{\sqrt{5}+1}{2}\right)^2\right]$ isn't a range of $|z|$ because in fact $\left[\begin{array}{l} \left(\dfrac{\sqrt{5}-1}{2}\right)^2 \le |z| \le \dfrac{\sqrt{5}-1}{2} \\ \dfrac{\sqrt{5}+1}{2} \le |z| \le \left(\dfrac{\sqrt{5}+1}{2}\right)^2 \end{array}\right.$.

$\Diamond$

J358. Proposed by Titu Andreescu, University of Texas at Dallas, USA

Prove that for $x \in \mathbb{R}$, the equations,
$2^{2^{x-1}} = \dfrac{1}{2^{2^x} - 1}$ and $2^{2^{x+1}} = \dfrac{1}{2^{2^{x-1}} - 1}$
are equivalent.

Solution by Arkady Alt , San Jose ,California, USA.

Let $t := 2^{2^{x-1}}$ then $t^2 = 2^{2 \cdot 2^{x-1}} = 2^{2^{x+1}}, t^4 = 2^{2^2 \cdot 2^{x-1}} = 2^{2^{x+1}}$ and equations can be rewritten,

respectively, as $t = \dfrac{1}{t^2 - 1} \iff t^3 - t - 1 = 0$ and $t^4 = \dfrac{1}{t-1} \iff t^5 - t^4 - 1 = 0$.

Since $t^5 - t^4 - 1 = \left(t^2 - t + 1\right)\left(t^3 - t - 1\right)$ and $t^2 - t + 1 \geq 3/4$ for any real t then

$t^5 - t^4 - 1 = 0 \iff t^3 - t - 1 = 0$, that is $t^4 = \dfrac{1}{t-1} \iff t = \dfrac{1}{t^2 - 1}$.

◊

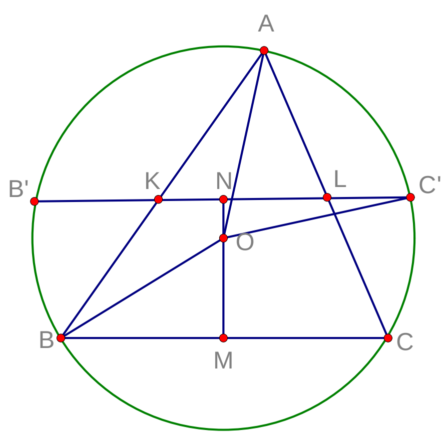
J359. Proposed by Dorin Andrica and Dan Ştefan Marinescu, România

The midline of triangle ABC, parallel to side BC, intersects the triangle's circumcircle at B' and C'.

Evaluate the length of segment $B'C'$ in terms of triangle ABC side-lengths.

Solution by Arkady Alt , San Jose ,California, USA.

First we will find distance ON from circumcenter to the midline.



Since $\angle OBC = 90^\circ - A$ then $OM = R \sin(90^\circ - A) = R \cos A$

Also since $MN = \dfrac{h_a}{2} = \dfrac{bc \sin A}{2a} = \dfrac{bc}{4a}$ then $ON = \dfrac{bc}{4a} - R \cos A$.

Therefore, $B'C' = 2\sqrt{R^2 - ON^2} = 2\sqrt{R^2 - \left(\dfrac{bc}{4R} - R\cos A\right)^2}$.

Since $\dfrac{bc}{4R} = \dfrac{abc}{4Ra} = \dfrac{F}{a}$ and $2bc \cos A = b^2 + c^2 - a^2$ then

$$R^2 - \left(\frac{bc}{4R} - R\cos A\right)^2 = R^2 \sin^2 A - \frac{1}{16R^2} b^2 c^2 + \frac{1}{2} bc \cos A =$$

$$\frac{a^2}{4} - \frac{F^2}{a^2} + \frac{b^2 + c^2 - a^2}{4} = \frac{b^2 + c^2}{4} - \frac{F^2}{a^2} =$$

$$\frac{b^2 + c^2}{4} - \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{16a^2} =$$

$$\frac{-2b^2c^2+2c^2a^2+2a^2b^2+a^4+b^4+c^4}{16a^2}=\frac{\left(a^2+(b+c)^2\right)\left(a^2+(b-c)^2\right)}{16a^2}$$

So, $B'C'=\dfrac{\sqrt{\left(a^2+(b+c)^2\right)\left(a^2+(b-c)^2\right)}}{2a}$.

◊

O355. Proposed by Nguyen Viet Hung, High School for Gifted Students, Hanoi University of Science, Vietnam

Let ABC be a triangle with incenter I. Prove that

$$\frac{(IB+IC)^2}{a(b+c)}+\frac{(IC+IA)^2}{b(c+a)}+\frac{(IA+IB)^2}{c(a+b)}\le 2.$$

Solution by Arkady Alt , San Jose ,California, USA.

Since $l_a=\dfrac{2\sqrt{bcs(s-a)}}{b+c},\dfrac{2\sqrt{cas(s-b)}}{c+a}$ then

$IA+IB=\dfrac{l_a(b+c)}{a+b+c}+\dfrac{l_b(c+a)}{a+b+c}=\dfrac{\sqrt{bcs(s-a)}}{s}+\dfrac{\sqrt{cas(s-b)}}{s}=$
$\sqrt{\dfrac{c}{s}}\left(\sqrt{b(s-a)}+\sqrt{a(s-b)}\right)$ and

$(IA+IB)^2=\dfrac{c}{s}\left(\sqrt{b(s-a)}+\sqrt{a(s-b)}\right)^2$.

By Cauchy Inequality

$\left(\sqrt{b(s-a)}+\sqrt{a(s-b)}\right)^2\le(b+a)(s-a+s-b)=c(a+b)$.

Thus, $(IA+IB)^2\le\dfrac{c^2}{s}(a+b)\iff\dfrac{(IA+IB)^2}{c(a+b)}\le\dfrac{c}{s}$ and, therefore,

$\sum\limits_{cyc}\dfrac{(IA+IB)^2}{c(a+b)}\le\sum\limits_{cyc}\dfrac{c}{s}=2.$

It is nice inequality!

◊

O357. Proposed by Titu Andreescu, USA and Oleg Mushkarov, Bulgaria

Prove that in any triangle

$\dfrac{ab+4m_am_b}{c}+\dfrac{bc+4m_bm_c}{a}+\dfrac{ca+4m_cm_a}{b}\ge\dfrac{16K}{R}$

Solution by Arkady Alt , San Jose ,California, USA.

Noting that $\dfrac{16K}{R}=\dfrac{64K^2}{4RK}=\dfrac{64K^2}{abc}$ and $xm_x\ge xh_x=2K, x\in\{a,b,c\}$ we obtain

$\sum\limits_{cyc}\dfrac{ab+4m_am_b}{c}=\sum\limits_{cyc}\dfrac{a^2b^2+4abm_am_b}{abc}\ge\dfrac{1}{abc}\sum\limits_{cyc}\left(a^2b^2+16K^2\right)$.

Thus suffice to prove inequality $\sum\limits_{cyc}\left(a^2b^2+16K^2\right)\ge 64K^2\iff$

(1) $\quad a^2b^2+b^2c^2+c^2a^2\ge 16K^2$.

Since $16K^2=2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$ then

(1) $\iff a^2b^2+b^2c^2+c^2a^2\ge 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4\iff$

$a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2 \iff \sum_{cyc}\left(a^2-b^2\right)^2 \geq 0.$

◊

S357. Proposed by Mihály Bencze, Braşov, România

Prove that in any triangle,

$$\sum_{cyc}\sqrt{\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)}} \leq \frac{3}{4}.$$

Solution by Arkady Alt , San Jose ,California, USA.

Since $h_a=\dfrac{2sr}{a}$ then $\dfrac{h_a-2r}{h_a+2r}=\dfrac{\dfrac{2sr}{a}-2r}{\dfrac{2sr}{a}+2r}=\dfrac{s-a}{s+a}$ and

$$\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)}=\frac{a\left(s-a\right)}{\left(2a+2s\right)\left(s+a\right)}=\frac{a\left(s-a\right)}{2\left(s+a\right)^2}=$$

$$\frac{2a\left(s-a\right)}{4\left(s+a\right)^2} \leq \frac{\left(\dfrac{2a+\left(s-a\right)}{2}\right)^2}{4\left(s+a\right)^2}=\frac{1}{16}.$$

Hence, $\displaystyle\sum_{cyc}\sqrt{\frac{a\left(h_a-2r\right)}{\left(3a+b+c\right)\left(h_a+2r\right)}} \leq \sum_{cyc}\sqrt{\frac{1}{16}}=\frac{3}{4}.$

◊

S359. Proposed by Titu Andreescu, University of Texas at Dallas, USA

Prove that in any triangle,

$$m_a\left(\frac{1}{2r_a}-\frac{R}{bc}\right)+m_b\left(\frac{1}{2r_b}-\frac{R}{ca}\right)+m_c\left(\frac{1}{2r_c}-\frac{R}{ab}\right) \geq 0.$$

Solution by Arkady Alt , San Jose ,California, USA.

Let F and s be area and semiperimeter of a triangle, respectively.

Since $r_a=\dfrac{F}{s-a}$ and $\dfrac{R}{bc}=\dfrac{aR}{abc}=\dfrac{aR}{4RF}=\dfrac{a}{4F}$ then

$$\frac{1}{2r_a}-\frac{R}{bc}=\frac{1}{4F}\left(2\left(s-a\right)-a\right)=\frac{1}{4F}\left(b+c-2a\right).$$

Therefore, $\displaystyle\sum_{cyc}m_a\left(\frac{1}{2r_a}-\frac{R}{bc}\right) \geq 0 \iff \sum_{cyc}m_a\left(b+c-2a\right) \geq 0 \iff$

(1) $\displaystyle\sum_{cyc}m_a\left(b+c\right) \geq 2\sum_{cyc}m_a a.$

Since $m_a^2-m_b^2=\dfrac{\left(2\left(b^2+c^2\right)-a^2\right)-\left(2\left(c^2+a^2\right)-b^2\right)}{4}=\dfrac{3\left(b-a\right)\left(a+b\right)}{4}$

yields $\left(m_a^2-m_b^2\right)\left(b-a\right)=\dfrac{3\left(b-a\right)^2\left(a+b\right)}{4} \geq 0$

then by rearrengement inequality for triples

$\left(m_a,m_b,m_c\right)$ and $\left(-a,-b,-c\right)$ holds inequalities

$\displaystyle\sum_{cyc}m_a\left(-a\right) \geq \sum_{cyc}m_a\left(-b\right) \iff \sum_{cyc}m_a b \geq \sum_{cyc}m_a a$

and $\displaystyle\sum_{cyc}m_a\left(-a\right) \geq \sum_{cyc}m_a\left(-c\right) \iff \sum_{cyc}m_a c \geq \sum_{cyc}m_a a.$

Thus, $\displaystyle\sum_{cyc}m_a\left(b+c\right)=\sum_{cyc}m_a b+\sum_{cyc}m_a c \geq 2\sum_{cyc}m_a a.$

◊

U357. Proposed by Dorin Andrica, Babes-Bolyai University, Cluj Napoca, România

Evaluate

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n$$

Solution by Arkady Alt , San Jose ,California, USA.

Let $a_n := \ln\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n =$

$n\left(\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)-\frac{1}{2}\right), n\in\mathbb{N}.$

Since $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right)$ then

$\ln\left(1+\frac{k}{n^2}\right)=\frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}+o\left(\frac{1}{n^3}\right)$ for $k=1,2,...,n.$

Hence, $a_n=n\left(\sum_{k=1}^{n}\left(\frac{k}{n^2}-\frac{k^2}{2n^4}+\frac{k^3}{3n^6}+o\left(\frac{1}{n^3}\right)\right)-\frac{1}{2}\right)=$

$\sum_{k=1}^{n}\left(\frac{k}{n}-\frac{k^2}{2n^3}+\frac{k^3}{3n^5}+no\left(\frac{1}{n^3}\right)\right)-\frac{n}{2}=$

$\sum_{k=1}^{n}\frac{k}{n}-\frac{n}{2}-\sum_{k=1}^{n}\frac{k^2}{2n^3}+\sum_{k=1}^{n}\frac{k^3}{3n^5}+o\left(\frac{1}{n^2}\right)=$

$\frac{n(n+1)}{2n}-\frac{n}{2}-\frac{n(n+1)(2n+1)}{12n^3}+\frac{n^2(n+1)^2}{12n^5}+o\left(\frac{1}{n^2}\right)=$

$\frac{1}{2}-\frac{(n+1)(2n+1)}{12n^2}+o\left(\frac{1}{n}\right)$ and, therefore,

$$\lim_{n\to\infty}a_n=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}.$$

Since $\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n=e^{a_n}$ then

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)...\left(1+\frac{n}{n^2}\right)}{\sqrt{e}}\right)^n=e^{\lim_{n\to\infty}a_n}=\sqrt[3]{e}.$$

◊

U358. Proposed by Mihai Piticari and Sorin Rădulescu, România.

Let $(x_n)_{n\geq 0}$ be an increasing sequence of real numbers for which there is a real number $a>2$ such that

$$x_{n+1} \geq ax_n - (a-1)x_{n-1},$$
for all $n \geq 1$. Prove that $(x_n)_{n\geq 1}$ is divergent.
Remark.
There is no need to claim that "$(x_n)_{n\geq 1}$ be an increasing sequence".
Suffice to claim $x_2 > x_1$.
Solution by Arkady Alt , San Jose ,California, USA.
$x_{n+1} \geq ax_n - (a-1)x_{n-1} \iff x_{n+1} - x_n \geq (a-1)(x_n - x_{n-1}) \iff$
$\dfrac{x_{n+1} - x_n}{(a-1)^n} \geq \dfrac{x_n - x_{n-1}}{(a-1)^{n-1}}, n \geq 1.$
Since $\left(\dfrac{x_{n+1} - x_n}{(a-1)^n}\right)_{n\geq 0}$ is increasing sequence then
$\dfrac{x_{n+1} - x_n}{(a-1)^n} \geq \dfrac{x_1 - x_0}{(a-1)^0} \implies x_{n+1} - x_n \geq (a-1)^n (x_1 - x_0) \implies$
$x_{n+1} - x_0 = \sum_{k=0}^{n} (x_{k+1} - x_k) \geq \sum_{k=0}^{n} (a-1)^k (x_2 - x_1) \iff$
$x_{n+1} - x_0 \geq \dfrac{(x_1 - x_0)\left((a-1)^n - 1\right)}{a-2}.$
Since $a > 2$ then by Bernoulli Inequality
$(a-1)^n \geq 1 + n(a-2), n \in \mathbb{N}$ and, therefore,
$x_{n+1} \geq x_0 + (x_1 - x_0)n, \ n \in \mathbb{N}.$
Thus, $(x_n)_{n\geq 1}$ is divergent.